Physics 211C: Solid State Physics

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Lecture 19

Topic: MZM in p+ip SC, Quantum hall

Vortices in SC:-

$$\mathcal{H}_{BCS \text{ mean field}} = \epsilon_k c_k^+ c_k + \Delta_k c_{k\uparrow}^+ c_{-k\downarrow}^+ + \Delta_k^* c_{-k\downarrow} c_{k\uparrow}$$

$$\Delta_k = k_x + i k_y \approx |k_F| e^{i\theta_k}$$

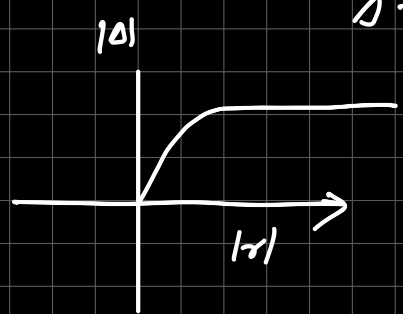
BdG eqns

$$E\, u(r) = -\mu(r)\, u(r) + \Delta^* \left[\partial_x + i\partial_y\right] v(r)$$

$$E\, v(r) = \mu(r)\, v(r) + \Delta(r)\, i\left[\partial_x - i\partial_y\right] u(r)$$

vortex: $\Delta(r) = |\Delta| e^{i\theta} \to$ winding 1

$\Delta \to 0$ at core of vortex



what is $\mu(r)$?

$\left.\begin{array}{l} \mu \to \text{large \& +ve in metal} \\ \to \text{" " -ve in SC} \end{array}\right\}$ why?

$\left.\begin{array}{l} u(\theta + 2\pi) = -u \\ v(\theta + 2\pi) = -v \end{array}\right\}$ why??? A toric code explanation.

$\mu(r \ggg 1) = -\mu_0 \quad \mu(r \lll 1) = +\mu_0$

Claim:- $\exists$ a sol^n s.t. $u = v^*$ $\begin{bmatrix} u \\ u^* \end{bmatrix} \rightarrow$ real fermion

$\partial_x - i\partial_y = 2\partial_z \quad (z = x+iy)$

for $E=0$

$$\left.\begin{aligned} -\mu u + 2|\Delta| i\partial_{\bar{z}} v &= 0 \\ \mu v + 2|\Delta| i\partial_z u &= 0 \end{aligned}\right\}$$ 2nd order eq^{n}s

reduct^n to 1st order

$\Downarrow$

has to do with supersymmetry

$\mu = v^*$

Use the ansatz

$$u = e^{i\gamma} (\bar{z})^{\alpha} f(z\bar{z})$$

$$\Downarrow$$

$$\gamma = -\frac{\pi}{4} \quad \alpha = -\frac{1}{2}$$

$$\mu f = -2|\Delta| \frac{df}{dr} \qquad f \sim e^{-\int_0^r \frac{|\Delta|(r')}{\mu(r')} dr'}$$

Ref:- Read & Greene (2000)

$\underbrace{\mu > 0 \ \& \ \mu < 0}$

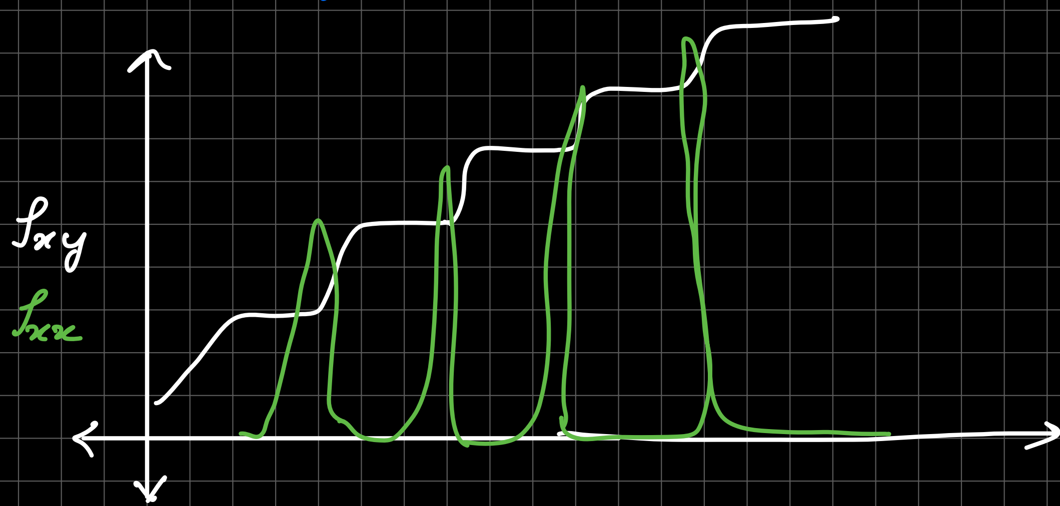
weak pairing phase vs strong pairing phase.

CHECK EXPLANATIONS

Quantum Hall

Ref:- Girvin & Yang, Shankar's book

Phenomenology :-



Within a plateau

$$\sigma_{xy} = \frac{\nu e^2}{h} \qquad \nu = \text{filling (fractional)}$$

when $\underbrace{\rho_{xy} \neq 0 , \ \rho_{xx} = 0 \Rightarrow \sigma_{xx} = 0}_{\text{no dissipation, yet no conduct}^n}$ (when $\rho \to 0$, we expect $\sigma \to \infty$)

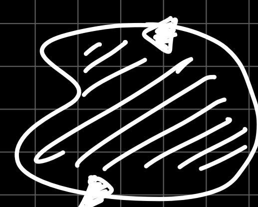
$\rho^{-1} = \sigma$

$$\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{xx} \end{bmatrix}^{-1}$$

(ii) Incompressible: density is indpt of chem pot.

(iii) Qps with fractional charge & anyonic statistics

(iv) Chiral edge modes
(also true for int q.h.)



all this explained by the low energy theory

$$S = i \int d^2x \, dt \left(\frac{k}{4\pi}\right) \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

where $k \in \mathbb{Z}$,

$$Z = e^{iS} \quad \longrightarrow \quad t \to i\tau \text{ doesn't do much.}$$

"Topological term" generally has this feature.

Zee's "what else could it be" argument :-

- $$\vec{J} = \frac{\vec{\nabla} \times \vec{a}}{2\pi}$$
- TRS is broken : $a_0 \xrightarrow{\tau} a_0$, $\vec{a} \xrightarrow{\tau} -\vec{a}$
- purely a topological term i.e. $H = 0$.

$$\mathcal{L} = \frac{k}{4\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + e A_\mu \underbrace{\frac{\epsilon_{\mu\nu\lambda}}{2\pi} \partial_\nu a_\lambda}_{\vec{A} \cdot \vec{J}} \text{ term.}$$

<u>E.O.M for a:</u>

$$\frac{k}{4\pi} a \wedge da = \frac{k\, a\, da}{4\pi}$$

$$\mathcal{L} = \frac{k}{4\pi} a\, da + e \frac{A\, da}{2\pi} = e \frac{a\, dA}{2\pi}$$

EOM:- $\frac{k}{4\pi} \cdot 2\, da + \frac{e\, dA}{2\pi} = 0$

$$\Rightarrow \quad da = -\frac{e\, dA}{k}$$

$$\overset{?}{\Rightarrow} \quad a = -\frac{eA}{k}$$

[assumption: no non-trivial holonomy of these objects]

$$J = \frac{\delta \mathcal{L}}{\delta A} = \frac{e\, da}{2\pi} = \frac{-e^2\, dA}{2\pi k}$$

$$J_0 = \frac{-e^2}{2\pi k} \underbrace{(\partial_x A_y - \partial_y A_x)}_{= B}$$

$\hookrightarrow$ density depends on $\vec{B}$ & not on μ

$\Downarrow$

Incompressible

$$J_x = \frac{-e^2}{2\pi k} - \partial_t A_y = \frac{e^2}{2\pi k} E_y$$

} reason for $\sigma_{xx} = 0 = \rho_{xx}$

see $\vec{J}$ & $\vec{E}$ are $\perp^r$ in responses.

$$\sigma_{xy} = \frac{e^2}{2\pi k} = \text{fractional hall conductance}$$

Mass gap for a :-

$$\mathcal{L} = -\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} + \frac{k}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon_{\mu\nu\lambda}$$

$$\partial_\mu f^{\mu\nu} = \frac{k}{4\pi} e^2 \epsilon^{\nu\rho\lambda} f_{\rho\lambda}$$

$$\tilde{f}_\mu = \epsilon_{\mu\nu\lambda} f^{\nu\lambda}$$

* operation for differential forms

$$\Big[\partial_\mu \partial \mu + \underbrace{\left(\frac{ke^2}{2\pi}\right)^2}_{\text{mass generated}} \Big] \tilde{f}_0 = 0$$

$$m = \frac{ke^2}{2\pi}$$

mass generation:- ① Higgs mechanism.

$(\nabla\phi - a)^2$ & condense ϕ

② Chern simons mass gap generation

$$\int d^2x\, d\tau\ a\,da + \int \underbrace{d^2x\, d\tau\, (da)^2}_{L}$$

must be irrelevant

$$aa \sim \frac{1}{L^2}$$

$$\propto \frac{1}{L^3} \longrightarrow \text{irrelevant}$$

Why is the smallest value of spin $\frac{1}{2}$? & not $\frac{1}{3}$rd?

why k?

$$e^{ik\int \frac{a\,da}{4\pi}} \qquad a_0 \to a_0 + \frac{d\theta}{dt}$$

$$\int \dot{\theta}\, dt = 2\pi n \qquad [a_x, a_y] = i$$

$e^{ik \times \text{flux integer}}$

Ref:- Flanders diff. forms for physics

$$\frac{a\,da}{4\pi} \xrightarrow[a]{\text{der. wrt}} \frac{2\,da}{4\pi}$$

Because of this kind of striking behaviour, the quantum Hall effect has been a constant source of new ideas, providing hints of where to look for interesting and novel phenomena, most of them related to the ways in which the mathematics of topology impinges on quantum physics. Important examples include the subject of topological insulators, topological order and topological quantum computing. All of them have their genesis in the quantum Hall effect.

Underlying all of these phenomena is an impressive theoretical edifice, which involves a tour through some of the most beautiful and important developments in theoretical and mathematical physics over the past decades. The first attack on the problem focussed on the microscopic details of the electron wavefunctions. Subsequent approaches looked at the system from a more coarse-grained, field-theoretic perspective where a subtle construction known as Chern-Simons theory plays the key role. Yet another perspective comes from the edge of the sample where certain excitations live that know more about what's happening inside than you might think. The main purpose of these lectures is to describe these different approaches and the intricate and surprising links between them.

1.2 The Classical Hall Effect

The original, classical Hall effect was discovered in 1879 by Edwin Hall. It is a simple consequence of the motion of charged particles in a magnetic field. We'll start these lectures by reviewing the underlying physics of the Hall effect. This will provide a useful background for our discussion of the quantum Hall effect.

Here's the set-up. We turn on a constant magnetic field, $\mathbf{B}$ pointing in the z-direction. Meanwhile, the electrons are restricted to move only in the (x, y)-plane. A constant current I is made to flow in the x-direction. The *Hall effect* is the statement that this induces a voltage V_H (H is for "Hall") in the y-direction. This is shown in the figure to the right.

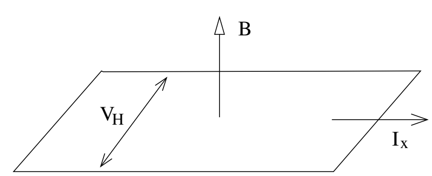


Figure 1: The classical Hall effect

1.2.1 Classical Motion in a Magnetic Field

The Hall effect arises from the fact that a magnetic field causes charged particles to move in circles. Let's recall the basics. The equation of motion for a particle of mass m and charge $-e$ in a magnetic field is

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{v} \times \mathbf{B}$$

When the magnetic field points in the z-direction, so that $\mathbf{B} = (0, 0, B)$, and the particle moves only in the transverse plane, so $\mathbf{v} = (\dot{x}, \dot{y}, 0)$, the equations of motion become two, coupled differential equations

$$m\ddot{x} = -eB\dot{y} \quad \text{and} \quad m\ddot{y} = eB\dot{x} \tag{1.1}$$

The general solution is

$$x(t) = X - R\sin(\omega_B t + \phi) \quad \text{and} \quad y(t) = Y + R\cos(\omega_B t + \phi) \tag{1.2}$$

We see that the particle moves in a circle which, for $B > 0$, is in an anti-clockwise direction. The centre of the circle, (X, Y), the radius of the circle R and the phase ϕ are all arbitrary. These are the four integration constants from solving the two second order differential equations. However, the frequency with which the particle goes around the circle is fixed, and given by

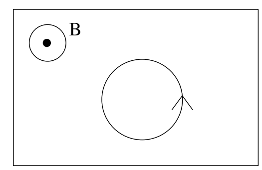


Figure 2:

$$\omega_B = \frac{eB}{m} \tag{1.3}$$

This is called the *cyclotron frequency*.

1.2.2 The Drude Model

Let's now repeat this calculation with two further ingredients. The first is an electric field, $\mathbf{E}$. This will accelerate the charges and, in the absence of a magnetic field, would result in a current in the direction of $\mathbf{E}$. The second ingredient is a linear friction term, which is supposed to capture the effect of the electron bouncing off whatever impedes its progress, whether impurities, the underlying lattice or other electrons. The resulting equation of motion is

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - \frac{m\mathbf{v}}{\tau} \tag{1.4}$$

The coefficient τ in the friction term is called the *scattering time*. It can be thought of as the average time between collisions.

The equation of motion (1.4) is the simplest model of charge transport, treating the mobile electrons as if they were classical billiard balls. It is called the *Drude model* and we met it already in the lectures on *Electromagnetism*.

We're interested in equilibrium solutions of (1.4) which have $d\mathbf{v}/dt = 0$. The velocity of the particle must then solve

$$\mathbf{v} + \frac{e\tau}{m}\mathbf{v} \times \mathbf{B} = -\frac{e\tau}{m}\mathbf{E} \tag{1.5}$$

The current density $\mathbf{J}$ is related to the velocity by

$$\mathbf{J} = -ne\mathbf{v}$$

where n is the density of charge carriers. In matrix notation, (1.5) then becomes

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}\mathbf{J} = \frac{e^2n\tau}{m}\mathbf{E}$$

We can invert this matrix to get an equation of the form

$$\mathbf{J} = \sigma\mathbf{E}$$

This equation is known as *Ohm's law*: it tells us how the current flows in response to an electric field. The proportionality constant σ is the *conductivity*. The slight novelty is that, in the presence of a magnetic field, σ is not a single number: it is a matrix. It is sometimes called the *conductivity tensor*. We write it as

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} \tag{1.6}$$

The structure of the matrix, with identical diagonal components, and equal but opposite off-diagonal components, follows from rotational invariance. From the Drude model, we get the explicit expression for the conductivity,

$$\sigma = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix} \quad \text{with} \quad \sigma_{DC} = \frac{ne^2\tau}{m}$$

Here σ_{DC} is the DC conductivity in the absence of a magnetic field. (This is the same result that we derived in the *Electromagnetism* lectures). The off-diagonal terms in the matrix are responsible for the Hall effect: in equilibrium, a current in the x-direction requires an electric field with a component in the y-direction.

Although it's not directly relevant for our story, it's worth pausing to think about how we actually approach equilibrium in the Hall effect. We start by putting an electric field in the x-direction. This gives rise to a current density J_x, but this current is deflected due to the magnetic field and bends towards the y-direction. In a finite material, this results in a build up of charge along the edge and an associated electric field E_y. This continues until the electric field E_y cancels the bending of due to the magnetic field, and the electrons then travel only in the x-direction. It's this induced electric field E_y which is responsible for the Hall voltage V_H.

Resistivity vs Resistance

The *resistivity* is defined as the inverse of the conductivity. This remains true when both are matrices,

$$\rho = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix} \tag{1.7}$$

From the Drude model, we have

$$\rho = \frac{1}{\sigma_{DC}} \begin{pmatrix} 1 & \omega_B \tau \\ -\omega_B \tau & 1 \end{pmatrix} \tag{1.8}$$

The off-diagonal components of the resistivity tensor, $\rho_{xy} = \omega_B \tau / \sigma_{DC}$, have a couple of rather nice properties. First, they are independent of the scattering time τ. This means that they capture something fundamental about the material itself as opposed to the dirty messy stuff that's responsible for scattering.

The second nice property is to do with what we measure. Usually we measure the *resistance* R, which differs from the resistivity ρ by geometric factors. However, for ρ_{xy}, these two things coincide. To see this, consider a sample of material of length L in the y-direction. We drop a voltage V_y in the y-direction and measure the resulting current I_x in the x-direction. The transverse resistance is

$$R_{xy} = \frac{V_y}{I_x} = \frac{LE_y}{LJ_x} = \frac{E_y}{J_x} = -\rho_{xy}$$

This has the happy consequence that what we calculate, ρ_{xy}, and what we measure, R_{xy}, are, in this case, the same. In contrast, if we measure the longitudinal resistance R_{xx} then we'll have to divide by the appropriate lengths to extract the resistivity ρ_{xx}. Of course, these lectures are about as theoretical as they come. We're not actually going to measure anything. Just pretend.

While we're throwing different definitions around, here's one more. For a current I_x flowing in the x-direction, and the associated electric field E_y in the y-direction, the *Hall coefficient* is defined by

$$R_H = -\frac{E_y}{J_x B} = \frac{\rho_{xy}}{B}$$

So in the Drude model, we have

$$R_H = \frac{\omega_B}{B\sigma_{DC}} = \frac{1}{ne}$$

As promised, we see that the Hall coefficient depends only on microscopic information about the material: the charge and density of the conducting particles. The Hall coefficient does not depend on the scattering time τ; it is insensitive to whatever friction processes are at play in the material.

We now have all we need to make an experimental prediction! The two resistivities should be

$$\rho_{xx} = \frac{m}{ne^2\tau} \quad \text{and} \quad \rho_{xy} = \frac{B}{ne}$$

Note that only ρ_{xx} depends on the scattering time τ, and $\rho_{xx} \to 0$ as scattering processes become less important and $\tau \to \infty$. If we plot the two resistivities as a function of the magnetic field, then our classical expectation is that they should look the figure on the right.

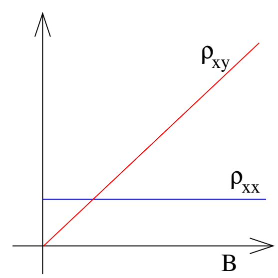


Figure 3:

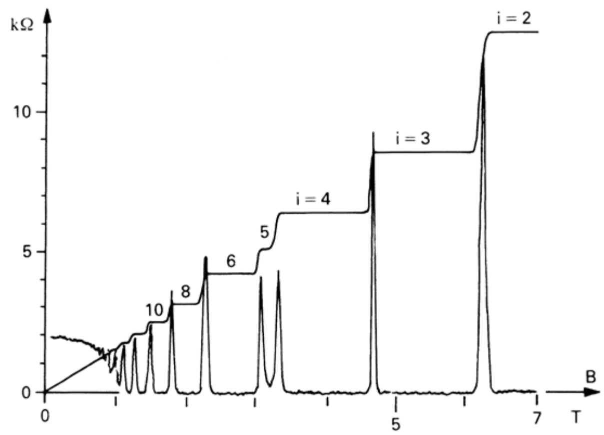
1.3 Quantum Hall Effects

Now we understand the classical expectation. And, of course, this expectation is borne out whenever we can trust classical mechanics. But the world is governed by quantum mechanics. This becomes important at low temperatures and strong magnetic fields where more interesting things can happen.

It's useful to distinguish between two different quantum Hall effects which are associated to two related phenomena. These are called the *integer* and *fractional* quantum Hall effects. Both were first discovered experimentally and only subsequently understood theoretically. Here we summarise the basic facts about these effects. The goal of these lectures is to understand in more detail what's going on.

1.3.1 Integer Quantum Hall Effect

The first experiments exploring the quantum regime of the Hall effect were performed in 1980 by von Klitzing, using samples prepared by Dorda and Pepper[1]. The resistivities look like this:



This is the *integer quantum Hall effect*. For this, von Klitzing was awarded the 1985 Nobel prize.

Both the Hall resistivity ρ_{xy} and the longitudinal resistivity ρ_{xx} exhibit interesting behaviour. Perhaps the most striking feature in the data is the fact that the Hall resistivity ρ_{xy} sits on a plateau for a range of magnetic field, before jumping suddenly to the next plateau. On these plateau, the resistivity takes the value

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \quad \nu \in \mathbf{Z} \tag{1.9}$$

The value of ν is measured to be an integer to an extraordinary accuracy. The quantity $2\pi\hbar/e^2$ is called the *quantum of resistivity* (with $-e$, the electron charge). It is now used as the standard for measuring of resistivity. Because ν is measured to be an integer to such remarkable precision – different devices differ only by 3 parts in 10^{10} – the integer quantum Hall effect is now used as the basis for measuring the ratio of fundamental constants $2\pi\hbar/e^2$ sometimes referred to as the von Klitzing constant[2]. This means that, by definition, the $\nu = 1$ state in (1.9) is exactly integer!

[1] K. v Klitzing, G. Dorda, M. Pepper, "*New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*", Phys. Rev. Lett. **45** 494.

[2] Full details of the different quantum Hall set ups and ways to measure the Hall resistivity can be found in B. Jeckelmann and B. Jeanneret, "*The quantum Hall effect as an electrical resistance standard*", Rep. Prog. Phys. 64, 1603-1655 (2001).

The centre of each of these plateaux occurs when the magnetic field takes the value

$$B = \frac{2\pi\hbar n}{\nu e} = \frac{n}{\nu}\Phi_0$$

where n is the electron density and $\Phi_0 = 2\pi\hbar/e$ is known as the *flux quantum*. As we will review in Section 2, these are the values of the magnetic field at which the first $\nu \in \mathbf{Z}$ Landau levels are filled. In fact, as we will see, it is very easy to argue that the Hall resistivity should take value (1.9) when ν Landau levels are filled. The surprise is that the plateau exists, with the quantisation persisting over a range of magnetic fields.

There is a clue in the experimental data about the origin of the plateaux. Experimental systems are typically dirty, filled with impurities. The technical name for this is *disorder*. Usually one wants to remove this dirt to get at the underlying physics. Yet, in the quantum Hall effect, as you increase the amount of disorder (within reason) the plateaux become more prominent, not less. In fact, in the absence of disorder, the plateaux are expected to vanish completely. That sounds odd: how can the presence of dirt give rise to something as exact and pure as an integer? This is something we will explain in Section 2.

The longitudinal resistivity ρ_{xx} also exhibits a surprise. When ρ_{xy} sits on a plateau, the longitudinal resistivity vanishes: $\rho_{xx} = 0$. It spikes only when ρ_{xy} jumps to the next plateau.

Usually we would think of a system with $\rho_{xx} = 0$ as a perfect conductor. But there's something a little counter-intuitive about vanishing resistivity in the presence of a magnetic field. To see this, we can return to the simple definition (1.7) which, in components, reads

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2} \quad \text{and} \quad \sigma_{xy} = \frac{-\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} \tag{1.10}$$

If $\rho_{xy} = 0$ then we get the familiar relation between conductivity and resistivity: $\sigma_{xx} = 1/\rho_{xx}$. But if $\rho_{xy} \neq 0$, then we have the more interesting relation above. In particular, we see

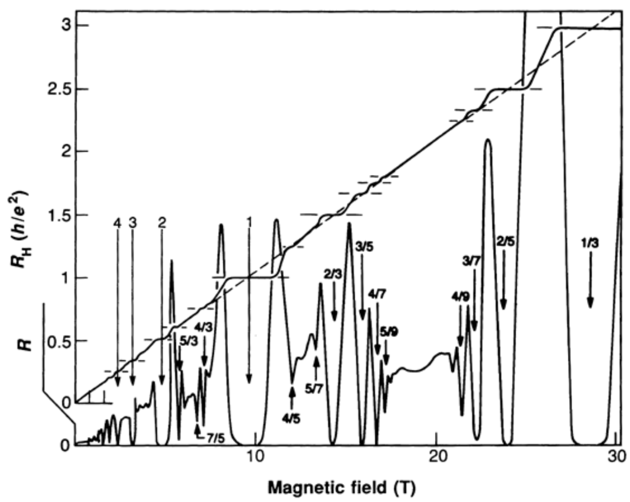
$$\rho_{xx} = 0 \quad \Rightarrow \quad \sigma_{xx} = 0 \qquad (\text{if } \rho_{xy} \neq 0)$$

While we would usually call a system with $\rho_{xx} = 0$ a perfect conductor, we would usually call a system with $\sigma_{xx} = 0$ a perfect insulator! What's going on?

This particular surprise has more to do with the words we use to describe the phenomena than the underlying physics. In particular, it has nothing to do with quantum mechanics: this behaviour occurs in the Drude model in the limit $\tau \to \infty$ where there is no scattering. In this situation, the current is flowing perpendicular to the applied electric field, so $\mathbf{E} \cdot \mathbf{J} = 0$. But recall that $\mathbf{E} \cdot \mathbf{J}$ has the interpretation as the work done in accelerating charges. The fact that this vanishes means that we have a steady current flowing without doing any work and, correspondingly, without any dissipation. The fact that $\sigma_{xx} = 0$ is telling us that no current is flowing in the longitudinal direction (like an insulator) while the fact that $\rho_{xx} = 0$ is telling us that there is no dissipation of energy (like in a perfect conductor).

1.3.2 Fractional Quantum Hall Effect

As the disorder is decreased, the integer Hall plateaux become less prominent. But other plateaux emerge at fractional values. This was discovered in 1982 by Tsui and Störmer using samples prepared by Gossard[3]. The resistivities look like this:



This is the *fractional quantum Hall effect*. On the plateaux, the Hall resistivity again takes the simple form (1.9), but now with ν a rational number

$$\nu \in \mathbf{Q}$$

Not all fractions appear. The most prominent plateaux sit at $\nu = 1/3, 1/5$ (not shown above) and $2/5$ but there are many more. The vast majority of these have denominators which are odd. But there are exceptions: in particular a clear plateaux has been observed at $\nu = 5/2$.

[3]D. C. Tsui, H. L. Stormer, and A. C. Gossard, "*Two-Dimensional Magnetotransport in the Extreme Quantum Limit*", Phys. Rev. Lett. **48** (1982)1559.